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SHORTER COMMUNICATIONS

PERTURBATION SOLUTION FOR PLANAR SOLIDIFICATION OF A SATURATED LIQUID WITH CONVECTION AT THE WALL

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NOMENCLATURE

c_p, heat capacity of frozen layer;
h, convective heat-transfer coefficient;
k, thermal conductivity of frozen solid;
L, latent heat of solidification;
T, temperature within frozen layer;
T_f, freezing temperature;
T_0, temperature of coolant;
t, time;
U, dimensionless temperature, $(T_f - T)/(T_f - T_0)$;
U_i, coefficient of ε^i in the power series expansion of U;
X, spatial position measured from the wall;
X_f, position of the moving interface;
x, dimensionless position, hX/k;
x_f, dimensionless thickness of frozen layer, hX_f/k.

Greek symbols

δ, immobilized distance, x/x_f;
ε, Stefan number, $c_p(T_f - T_0)/L$;
ρ, density of frozen layer;
τ, Fourier number, $h^2t/\rho c_p k$;
τ_i, coefficient of ε^i in the power series expansion of $\varepsilon\tau$.

INTRODUCTION

THE PLANAR solidification of a saturated liquid with convection at the wall has been discussed by Carslaw and Jaeger [1], Lock [2], Goodman [3], and Pedroso and Domoto [4]. Pedroso and Domoto [4] found a perturbation solution for this problem.

In this report, a perturbation solution is obtained by the use of a new method [5] of the authors. The new method consists of (1) immobilizing the moving interface by Landau transformation, (2) replacing the time variable by interface position as independent variable, and (3) applying the regular parameter perturbation technique. Landau transformation makes the nonlinearity due to moving interface explicit. However, the use of Landau transformation in the perturbation solutions for bubble growth was discussed by Duda and Vrentas [6]. Pedroso and Domoto [4] replaced the time variable by the interface position to obtain a perturbation for this problem and similar solidification problems [7]. Replacing time variable by the interface position was also used by Siegel and Savino [8] in finding the analytical iterative solutions of moving boundary problems.

ANALYSIS

For planar solidification of a saturated liquid with convection at the wall and constant physical properties of the frozen material, the temperature distribution $T(X, t)$ satisfies the transient heat-conduction equation.

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial X^2} \tag{1}$$

where t is the time variable, X is the spatial position measured from the wall, c_p, ρ and k are, respectively, the heat capacity, density and thermal conductivity of the frozen material. The boundary condition at the wall is of convective heat transfer with constant heat-transfer coefficient h,

$$k \left.\frac{\partial T}{\partial X}\right|_{X=0} = h[T(0, t) - T_0] \tag{2}$$

where T_0 is the temperature of the coolant. The temperature of frozen material at the moving interface, $X = X_f(t)$, equals the freezing temperature T_f

$$T(X_f, t) = T_f. \tag{3}$$

The energy balance at the moving interface gives

$$\rho L \frac{\mathrm{d}X_f}{\mathrm{d}t} = k \left.\frac{\partial T}{\partial X}\right|_{X=X_f} \tag{4}$$

where L is the latent heat of solidification. Assuming zero initial thickness of frozen material yields the initial condition of equation (4),

$$X_f(0) = 0. \tag{5}$$

Introducing the dimensionless variables

$$x = \frac{hX}{k}$$
$$x_f = \frac{hX_f}{k}$$
$$\tau = \frac{h^2 t}{\rho c_p k} \tag{6}$$
$$\varepsilon = \frac{c_p(T_f - T_0)}{L}$$
$$U = \frac{T_f - T}{T_f - T_0}.$$

Equations (1–5) become

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} \tag{7}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = U(0, \tau) - 1 \tag{8}$$

$$U(x_f, \tau) = 0 \tag{9}$$

$$\frac{\mathrm{d}x_f}{\mathrm{d}\tau} = -\varepsilon \left.\frac{\partial U}{\partial x}\right|_{x=x_f} \tag{10}$$

$$x_f(0) = 0. \tag{11}$$

Notice that $U = U(x, \tau)$ and $x_f = x_f(\tau)$.

The interface position is immobilized by using

$$\delta = \frac{x}{x_f} \tag{12}$$

as spatial variable. Transformation of $U(x, \tau)$ into $U(\delta, x_f)$ yields

$$\varepsilon\left(\delta\frac{\partial U}{\partial \delta}-x_f\frac{\partial U}{\partial x_f}\right)\left(\frac{\partial U}{\partial \delta}\bigg|_{\delta=1}\right)=\frac{\partial^2 U}{\partial \delta^2} \tag{13}$$

$$\frac{\partial U}{\partial \delta}\bigg|_{\delta=0}=x_f[U(0,x_f)-1] \tag{14}$$

$$U(1,x_f)=0 \tag{15}$$

$$\varepsilon\frac{\mathrm{d}\tau}{\mathrm{d}x_f}=-x_f\left(\frac{\partial U}{\partial \delta}\bigg|_{\delta=1}\right)^{-1} \tag{16}$$

$$\tau(x_f)=0, \quad \text{at} \quad x_f=0 \tag{17}$$

τ is thus expressed as function of x_f.

The regular parameter perturbation method is used with

$$U(\delta,x_f)=U_0(\delta,x_f)+\varepsilon U_1(\delta,x_f)+\varepsilon^2 U_2(\delta,x_f)+\varepsilon^3 U_3(\delta,x_f)+\dots \tag{18}$$

Substituting equation (18) into equations (13)–(15) and equating coefficients of equal powers of ε give

$$\frac{\partial^2 U_0}{\partial \delta^2}=0 \tag{19}$$

$$\frac{\partial^2 U_1}{\partial \delta^2}=\left(\delta\frac{\partial U_0}{\partial \delta}-x_f\frac{\partial U_0}{\partial x_f}\right)\left(\frac{\partial U_0}{\partial \delta}\bigg|_{\delta=1}\right) \tag{20}$$

$$\frac{\partial U_2}{\partial \delta^2}=\left(\delta\frac{\partial U_1}{\partial \delta}-x_f\frac{\partial U_1}{\partial x_f}\right)\left(\frac{\partial U_0}{\partial \delta}\bigg|_{\delta=1}\right)+\left(\delta\frac{\partial U_0}{\partial \delta}-x_f\frac{\partial U_0}{\partial x_f}\right)\left(\frac{\partial U_1}{\partial \delta}\bigg|_{\delta=1}\right) \tag{21}$$

$$\frac{\partial^2 U_3}{\partial \delta^2}=\left(\delta\frac{\partial U_2}{\partial \delta}-x_f\frac{\partial U_2}{\partial x_f}\right)\left(\frac{\partial U_0}{\partial \delta}\bigg|_{\delta=1}\right)+\left(\delta\frac{\partial U_1}{\partial \delta}-x_f\frac{\partial U_1}{\partial x_f}\right)\times\left(\frac{\partial U_1}{\partial \delta}\bigg|_{\delta=1}\right)+\left(\delta\frac{\partial U_0}{\partial \delta}-x_f\frac{\partial U_0}{\partial x_f}\right)\left(\frac{\partial U_2}{\partial \delta}\bigg|_{\delta=1}\right) \tag{22}$$

with boundary conditions

$$\frac{\partial U_0}{\partial \delta}\bigg|_{\delta=0}=x_f[U_0(0,x_f)-1] \tag{23}$$

$$\frac{\partial U_1}{\partial \delta}\bigg|_{\delta=0}=x_f U_1(0,x_f), \quad i=1,2,\dots \tag{24}$$

$$U_i(1,x_f)=0, \quad i=0,1,2,\dots. \tag{25}$$

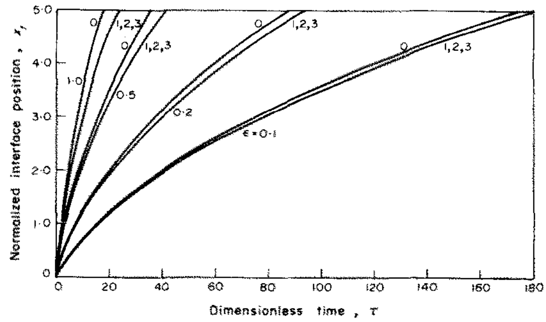

FIG. 1. Convergence of perturbations of freezing time: 0, $\varepsilon\tau=\tau_0$; 1, $\varepsilon\tau=\tau_0+\varepsilon\tau_1$; 2, $\varepsilon\tau=\tau_0+\varepsilon\tau_1+\varepsilon^2\tau_2$; 3, $\varepsilon\tau=\tau_0+\varepsilon\tau_1+\varepsilon^2\tau_2+\varepsilon^3\tau_3$.

The analytical solutions of equations (19–25) are

$$U_0=\frac{x_f}{1+x_f}(1-\delta) \tag{26}$$

$$U_1=\frac{x_f^2}{6(1+x_f)^4}[(1+x_f)(3+x_f\delta)\delta^2-(3+x_f)(1+x_f\delta)] \tag{27}$$

$$U_2(\delta,x_f)=-\frac{x_f^3}{360(1+x_f)^7}[9x_f(1+x_f)^2(5+x_f\delta)\delta^4+10(1+x_f)(12+3x_f+x_f^2)(3+x_f\delta)\delta^2-(360+225x_f+114x_f^2+19x_f^3)(1+x_f\delta)] \tag{28}$$

$$U_3=\frac{x_f^4}{15120(1+x_f)^{10}}\{45x_f^2(1+x_f^2)\times[(1+x_f)(7+x_f\delta)\delta^6-(7+x_f)(1+x_f\delta)]+63(1+x_f)(-4+24x_f+9x_f^2+3x_f^3)\times[(1+x_f)(5+x_f\delta)\delta^4-(5+x_f)(1+x_f\delta)]+7(2160+840x_f+525x_f^2+102x_f^3+17x_f^4)\times[(1+x_f)(3+x_f\delta)\delta^2-(3+x_f)(1+x_f\delta)]\}. \tag{29}$$

The position of the moving interface is also taken as a power series expansion of $\varepsilon\tau(x_f)$,

$$\varepsilon\tau(x_f)=\tau_0(x_f)+\varepsilon\tau_1(x_f)+\varepsilon^2\tau_2(x_f)+\varepsilon^3\tau_3(x_f)+\dots. \tag{30}$$

Table 1. First four terms of the freezing time

x_f	τ_0	τ_1	τ_2	τ_3	τ_3^*
0·0	0·0000	0·0000	0·0000	0·0000	0·0000
0·2	0·2200	0·01778	−0·001564	−0·0003954	0·0003945
0·4	0·4800	0·06476	−0·008154	−0·002868	0·002846
0·6	0·7800	0·1350	−0·01932	−0·007604	0·007488
0·8	1·120	0·2252	−0·03398	−0·01400	0·01368
1·0	1·500	0·3333	−0·05139	−0·02158	0·02092
1·2	1·920	0·4582	−0·07109	−0·03006	
1·4	2·380	0·5989	−0·09289	−0·03932	0·03768
1·6	2·880	0·7549	−0·01164	−0·04926	
1·8	3·420	0·9257	−0·1418	−0·05986	0·05692
2·0	4·000	1·1111	−0·1690	−0·07109	
2·2	4·620	1·3108	−0·1979	−0·08294	0·07854
2·4	5·280	1·525	−0·2285	−0·09542	
2·6	5·980	1·752	−0·2608	−0·1085	0·1025
2·8	6·720	1·994	−0·2948	−0·1222	
3·0	7·500	2·250	−0·3305	−0·1366	0·1292
4·0	12·00	3·733	−0·5348	−0·2177	0·2068
5·0	17·50	5·555	−0·7823	−0·3147	0·3008

*Values obtained by [4].

Substitution of equations (18) and (30) into equation (16) and equating the coefficients of equal powers of ε yield

$$\tau_0 = \int_0^{x_f} \left(\frac{\partial U_0}{\partial \delta}\bigg|_{\delta=1} \right)^{-1} x_f \, dx_f \quad (31)$$

$$\tau_1 = \int_0^{x_f} \left(\frac{\partial U_1}{\partial \delta}\bigg|_{\delta=1} \right)\left(\frac{\partial U_0}{\partial \delta}\bigg|_{\delta=1} \right)^{-2} x_f \, dx_f \quad (32)$$

$$\tau_2 = \int_0^{x_f} \left[\left(\frac{\partial U_0}{\partial \delta}\bigg|_{\delta=1} \right)\left(\frac{\partial U_2}{\partial \delta}\bigg|_{\delta=1} \right) - \left(\frac{\partial U_1}{\partial \delta}\bigg|_{\delta=1} \right)^2 \right]\left(\frac{\partial U_0}{\partial \delta}\bigg|_{\delta=1} \right)^{-3} x_f \, dx_f \quad (33)$$

$$\tau_3 = \int_0^{x_f} \left[2\left(\frac{\partial U_0}{\partial \delta}\bigg|_{\delta=1} \right)\left(\frac{\partial U_1}{\partial \delta}\bigg|_{\delta=1} \right)\left(\frac{\partial U_2}{\partial \delta}\bigg|_{\delta=1} \right) - \left(\frac{\partial U_0}{\partial \delta}\bigg|_{\delta=1} \right)^2\left(\frac{\partial U_3}{\partial \delta}\bigg|_{\delta=1} \right) - \left(\frac{\partial U_1}{\partial \delta}\bigg|_{\delta=1} \right)^3 \right]\left(\frac{\partial U_0}{\partial \delta} \bigg| \right)^{-4} x_f \, dx_f \quad (34)$$

where equation (17) has been used. Evaluation of τ_0, τ_1 and τ_2 by the use of equations (16)–(29) one obtains

$$\tau_0 = \tfrac{1}{2}[(1+x_f)^2 - 1] \quad (35)$$

$$\tau_1 = \frac{1}{6(1+x_f)}[(1+x_f)^3 - 3(1+x_f) + 2] \quad (36)$$

$$\tau_2 = \frac{-1}{45(1+x_f)^4}[(1+x_f)^6 - 5(1+x_f)^3 + 9(1+x_f) - 5] \quad (37)$$

$$\tau_3 = \frac{-1}{7560(1+x_f)^7}[64(1+x_f)^9 + 315(1+x_f)^7 - 2058(1+x_f)^6 + 4725(1+x_f)^5 - 6804(1+x_f)^4 + 4725(1+x_f)^3 + 1350(1+x_f)^2 - 3717(1+x_f) + 1400]. \quad (38)$$

Higher order solutions of U_i and τ_i may be obtained by the same procedure. However, algebraic manipulation is complicated.

RESULTS AND DISCUSSION

The effect of ε on the interface position is illustrated in Fig. 1. The departure from the quasi-steady state solution, i.e. zero-order solution, increases as Stefan number, ε, increases as well as x_f increases.

Table 1 shows the values of τ_0, τ_1, τ_2 and τ_3 for the values of normalized interface position up to $x_f = 5$. The values of τ_0, τ_1 and τ_2 are consistent with the result of Pedroso and Domoto [4]. The values of τ_3 are quite different from the values of τ_3 of [4], which are also listed in Table 1. The difference between the perturbation method of this communication and Pedroso and Domoto [4] method is the use of Landau transformation in this communication. Landau transformation makes the nonlinearity due to moving interface explicit. Therefore, perturbation method can be used in a straightforward manner.

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ON THE ANALYSIS OF CELLULAR CONVECTION IN POROUS MEDIA

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NOMENCLATURE

a, wave number;
a_0, critical wave number;
A, parameter defined in equation (17);
c, solute concentration (salinity);
$\bar{c}$, mean horizontal concentration;
d, porous layer thickness;
d_p, characteristic pore length;
g, gravitational acceleration;
H, solute advection spectrum;
$H_{pq}^{(n)}$, coefficient in the series expanded for H;
K, permeability;
N, number of terms in the series expanded for ψ and γ;
Pe, Peclet number (Ud_p/κ_s);
Re, Reynolds number (Ud_p/ν);
s, number of terms in the series expanded for S;
S, Rayleigh number $(\alpha_s g \Delta c K d/\nu\varepsilon\kappa_s)$;
S_0, critical Rayleigh number;
S_{0s}, parameter defined in equation (9);
Sc, Schmidt number (ν/κ_s);
U, module of velocity vector;
x, horizontal coordinate;
z, vertical coordinate.

Greek symbols
α_s, coefficient relating salinity with density;
γ, salinity perturbation;
$\Gamma_{pq}^{(n)}$, coefficient in the series expanded for γ;

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